PAL
10/19/07

Unified Engineering Problem Set
Week 9 Fall, 2007

SOLUTIONS

M9.1 Given:

$$\sigma_{12} = -A \; ; \quad \sigma_{13} = -B \; ; \quad \sigma_{23} = -C$$

NOTE: In most general case, all stresses are functions of all three dimensions:

$$\sigma_{mn} \, (x_1, x_2, x_3)$$

→ Use this information with the stress equations of equilibrium to determine information about the other stresses. First write the 3-D stress equilibrium equations.

→ Given: no body forces

So:
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0 \qquad (2)$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \qquad (3)$$

→ Take partial derivatives of what is known about stresses to give information

$$\frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} = 0$$

$$\frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{23}}{\partial x_2} = 0$$

- Use this information first in (1):

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0$$

using multi-variable calculus and getting functions of integration:

$$\sigma_{11} = C_1 + f_{11}(x_2, x_3)$$

$\Rightarrow \sigma_{11}$ is constant with respect to x_1 but some unknown function of x_2 and x_3

- use the information in (2):

$$\frac{\partial \sigma_{22}}{\partial x_2} = 0$$

again this gives an expression for σ_{22} via integration:

$$\sigma_{22} = C_2 + f_{22}(x_1, x_3)$$

- and then in (3):

$$\frac{\partial \sigma_{33}}{\partial x_3} = 0$$

giving....

$$\sigma_{33} = C_3 + f_{33}(x_1, x_2)$$

We thus end up with explicit constants for the three shear stresses (as given) and expressions for each extensional stress that is constant in one direction (an unknown value) and an unknown function ~~of~~ the other two directions.

This is summarized as:

$$\boxed{\begin{aligned}
\sigma_{12} &= -A \\
\sigma_{13} &= -B \\
\sigma_{23} &= -C \\
\sigma_{11} &= C_1 + f_{11}(x_2, x_3) \\
\sigma_{22} &= C_2 + f_{22}(x_1, x_3) \\
\sigma_{33} &= C_3 + f_{33}(x_1, x_2)
\end{aligned}}$$

M 9.2 We have a two-dimensional field of displacement with no displacement in the third (x_3) direction. Thus, all out-of-plane strains are zero as either u_3 is zero or $\frac{\partial}{\partial x_3} = 0$:

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$$

Now define the in-plane strain-displacement relations:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

with the displacement field defined as:

$$\underline{u} = u_1 \underline{i}_1 + u_2 \underline{i}_2$$

Apply this to each case to get the in-plane strains.

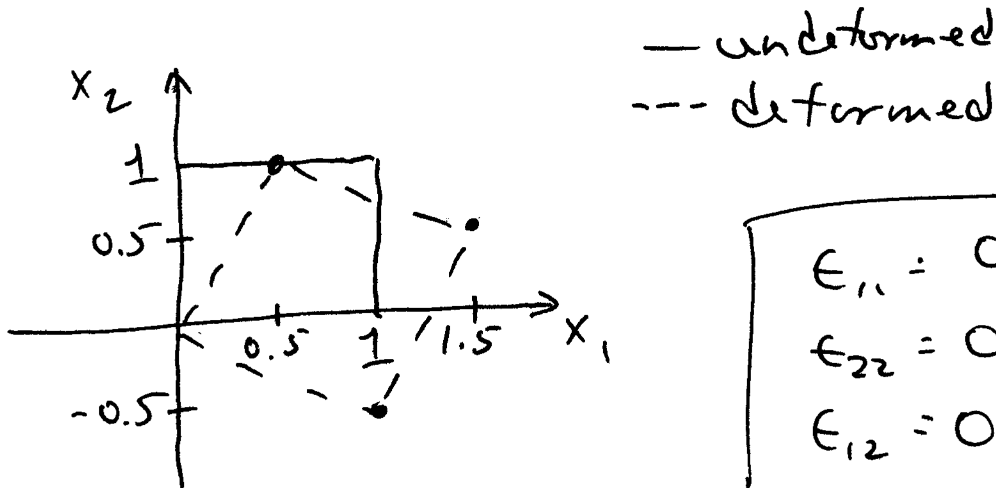
So:

(a) $\underline{u} = (0.050 x_2)\underline{i}_1 - (0.050 x_1)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0.050 - 0.050) = 0$$



$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$$

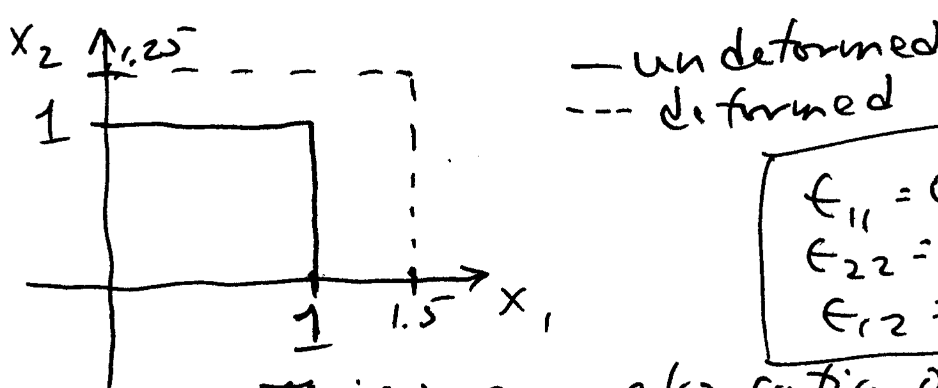
This is <u>pure</u> <u>rotation</u>

(b) $\underline{u} = (0.050 x_1)\underline{i}_1 + (0.025 x_2)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.050$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0.025$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = 0$$



$$\boxed{\begin{aligned} \epsilon_{11} &= 0.050 \\ \epsilon_{22} &= 0.025 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is <u>pure</u> <u>elongation</u> <u>in</u> <u>two</u> <u>directions</u>
(both positive)

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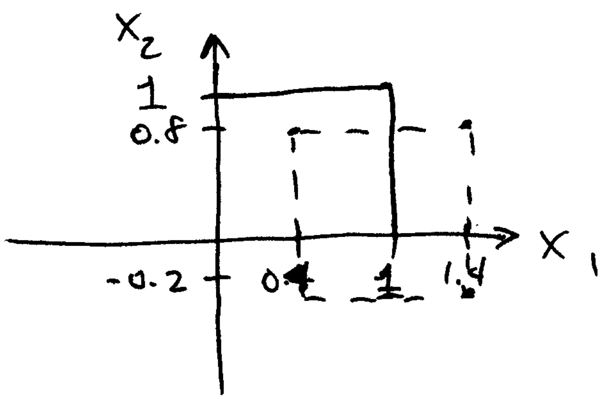
(c) $\underline{u} = (0.040)\underline{i}_1 - (0.020)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is <u>pure translation</u> in x_1 and x_2

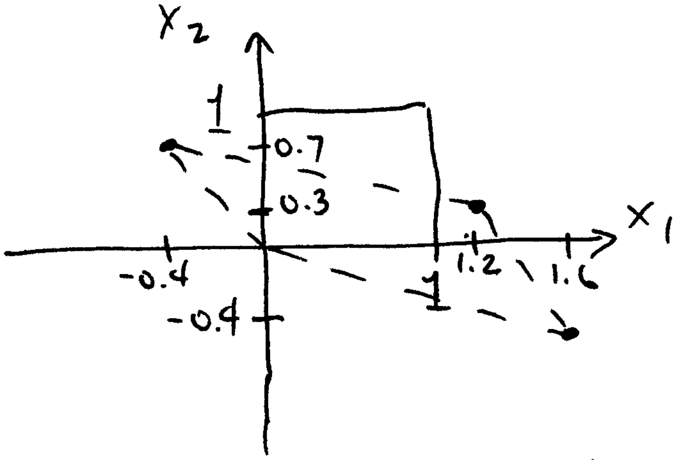
(d) $\underline{u} = (0.060x_1 - 0.040x_2)\underline{i}_1$
$\quad + (-0.040x_1 - 0.030x_2)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.060$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.030$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(-0.040 - 0.040)$$

$$= -0.040$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0.060 \\ \epsilon_{22} &= -0.030 \\ \epsilon_{12} &= -0.040 \end{aligned}}$$

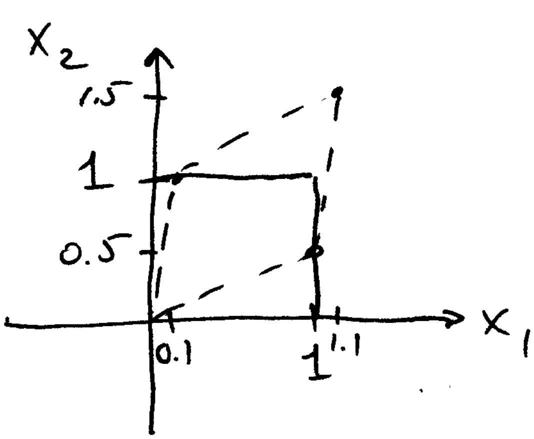
This is combined elongation and shear

(e) $\underline{u} = (0.010\,x_2)\,\underline{i}_1 + (0.050\,x_1)\,\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0.010 + 0.050)$$
$$= 0.030$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0.030 \end{aligned}}$$

This is pure shear